The Educator's Companion to Measurement in the Common Core State Standards in Mathematics (CCSS-M)
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The views expressed here, especially in this first draft from comment, are our own and have not yet been endorsed, checked for completeness, or critiqued by others.

Feedback Encouraged
This is a draft for comment. We encourage comments on its form and content from all who read it. We are particularly interested in how well it achieves its goal of unpacking the measurement content of the CCSS-M, calling attention to the measurement content is present (and absent) there, and relating the mathematical practices to measurement as one specific content strand. Comments on these issues (and any others) may be sent to stemproj@msu.edu or to either of the authors at, jsmith@msu.edu or gonulate@msu.edu.

Section I: Introduction

Purpose

This document was written to provide guidance to K-8 teachers and those who work directly with them in support of their instruction in mathematics and science in interpreting the measurement strand of the Common Core State Standards in Mathematics (CCSS-M). We hope to provide assistance with both the content standards and the mathematical practices, as the latter apply to measurement content. As we explain in more detail below, we have tried to provide a fair interpretive reading of the document's treatment of measurement—of both what is present and what is absent. We have sought language that would be accessible to teachers and accurate the mathematical and learning issues raised in and by the CCSS-M. Whether we have yet achieved, or even come close to that goal is another matter. Early readers can assist us by bringing these gaps and limitations to our attention.

Intended Use

We hope that this document will be useful in professional development and instructional support work by elementary educators through the United States, as states prepare for and begin to implement the CCSS-M. We also hope that it is useful to curriculum authors and serve as useful input for those who will be involved in subsequent revisions of the CCSS-M.

Why Measurement?

The Common Core State Standards in Mathematics address content and practices for all of pre-college school mathematics, Kindergarten through 12th grade for all content strands and topics. Why then select out one strand (or domain) of elementary mathematics for special attention? There are many reasons for the specific content focus of this document. First, the CCSS-M is too rich and broad in scope to address the goals stated above for all major content areas. The length of this document (addressing but one domain of elementary mathematics) is proof enough of that. Second, we are not prepared intellectually for that full task of interpretation; our specific expertise (and that of our mini-Center colleagues) lies in the domain of measurement, geometry, and topics usefully approached from the perspective of measurement. We think it best to work where we feel most prepared and hope that others carry out similar interpretive work in other content areas.

Third, it is too often the case that students and teachers' understandings of measurement are weak—in the case of teachers, because their own instruction in measurement was weak. The United States as a whole performs more poorly in measurement on the National Assessment of Educational Progress than in any of content domain assessed by NAEP (**Preston & Thompson, date**). The publication of the CCSS-M is therefore an excellent opportunity to re-envision the teaching of measurement on a stronger conceptual foundation than is currently the case in many of our nation's classrooms. The CCSS-M itself provides some of that conceptual grounding; this document has been written, in part, to fill out that grounding. Fourth, we believe, as do others, that a stronger foundation in measurement in the elementary grades will facilitate much deeper understanding of other "problematic" topics in other content areas. For example, we believe the meaning of rational numbers (especially when expressed as fractions) and operations on rational numbers become more accessible when those numbers are interpreted as continuous (not discrete) quantities, e.g., as lengths and areas. In that sense, this potential usefulness of this document is not confined to support for the teaching of measurement.

The Core of Measurement

In the most general terms, measurement involves the coordination of space and number. Given any one-dimensional, two-dimensional, or three-dimensional space, we measure that space when we identify a suitable unit (e.g., a two-dimensional unit for a two-dimensional space) and then apply that unit multiple times to "fill up" that space. The resulting measure number is the number of units required for that task, be it a whole number of units or a whole number of units and a fraction of that unit. In this way, measurement turns continuous quantities (here length, area, and volume) in discrete quantities, collections of units that we can count. This view of measurement is expressed in terms of one-, two-, and three-dimensional space, but the same basic process works for other quantities that we measure in school mathematics and science—such as time, weight, temperature, angle, density.

Measurement Content in the CCSS-M

Measurement content principally appears in the *Measurement and Data* domain in Grades K through 5 and in the *Geometry* domain in Grades 6 through 8. Most of the standards we list and interpret in this document are located in these two domains (K-8). However, because the linkage between geometry and measurement is so strong and important, especially for measurement in two and three dimensions, we also list and interpret standards from the *Geometry* domain, as needed, in Grades K through 5. In some grades, standards in the *Number and Operations-Fractions* domain are also listed and discussed, particularly for the treatment of fractions that is linked to partitioning area into equal-sized parts.

We note that CCSS-M authors have compiled all the *Measurement and Data* standards into one document (http://commoncoretools.files.wordpress.com/2011/05/measurement_and_data.pdf), but that is a very different document than the one we have prepared. Their document is simply a grade-by-grade listing of all standards that the authors have written and classified in the *Measurement and Data* domain, thus excluding all *Geometry* standards and introductions to each grade where specific foci are named. That document also provides no interpretive commentary on the standards and the introductory discussions.

The Focus and Structure of the Document

We have focused almost exclusively on spatial measurement in this document—that is, the measurement of length, area, and volume. The reasons for this focus are many, chief among them that the work of the STEM project has focused there and that spatial measurement is the content where U.S. mathematics (if not also science) curricula develop students' notions of measurement. We discuss briefly the inclusion of angle measurement when the document introduces it within "geometric measurement;" we do not consider or discuss the document's treatment of measurement of time, weight, temperature or any other quantity.

Overall, we pursue three major goals in communicating with educators who will be tasked to teach in the letter and spirit of the CCSS-M. First, we hope this document will make it easier to read and understand how measurement of length, area, and volume (respectively) are developed across grades, starting in Grade K. As stated above, our treatment of these topics is more comprehensive that the authors' listing of standards, grade by grade, in the *Measurement and Data* domain. In pursuing this descriptive goal, we quote introductory statements at each grade that refer to spatial measurement and then each specific standard that does so.

Second, we attempt an honest interpretation of the authors' text in language that we hope will be accessible and useful for teachers. This interpretation appears in two preliminary sections of the document and then at each grade level for each spatial measure. Prior to our discussion of length, area, and then volume measurement, we interpret the mathematical practices as we see that they connect with issues of measurement. Second, we discuss some key terms in the document's treatment of measurement that we think may not be familiar to elementary teachers. Then for each measure and in each grade, we present what we believe a fair reading of the authors' meaning in introductory text and specific standards and name issues where we cannot find a strong basis to offer such an interpretation. As with every text, the CCSS-M leaves some issues open, ambiguous, and/or unknown.

Third and equally important, we identify gaps in the document's treatment of spatial measurement, where issues that we feel should be addressed have not been. Chiefly, the discussion of missing content focuses on invisible and implicit conceptual principles, particularly for area and volume. We see two types of missing content: Issues that are measure-specific and issues that are common to all three spatial measures but are not addressed in all three.

Key Departures from Current State Standards & Elementary Mathematics Curricula

Though we do not offer an "executive summary" to highlight all the major ways that the CCSS-M departs from and will press on current curricular treatment and teaching practices for measurement, some major points are worthy of mention. Here we draw on our knowledge of existing treatments of spatial measurement in elementary textbooks and the content of current state standards for measurement (Kasten & Newton, 2011). We do not see major changes from current practice and curricula in the treatment of length measurement in the primary grades, though the current curricular repetition of content in subsequent grades is reduced. However, the CCSS-M's treatment of area and volume measurement is quite different than current practice. Most current curricula define area in Grade 2 and distribute attention across subsequent grades, where CCSS-M treats area seriously in Grade 3 and then inconsistently across grades (through Grade 8). To put it simply, the CCSS-M does more work with area in one particular grade than do current curricula. Current curricular treatments of capacity (liquid volume) begin in Grade K; volume is introduced via filling boxes with unit cubes shortly thereafter; and both measures receive some continuous attention through the middle school years. By contrast, save two mentions of "liquid volume" in prior grades, capacity appears nowhere in the document, and volume is not introduced until Grade 5. So, as was true for area, the CCSS-M does much more with volume at a single grade than do current curricular materials. Though we cannot infer typical classroom teaching practice from curricular content, both of these differences would seem to represent major changes from current practice.

Section II. Important Terms for Measurement in the CCSSM

We think some terms in the CCSS-M deserve special attention prior to examining the document's treatment of measurement. We have included this section to support educators' reading of the CCSS-M and to communicate what we see as the meanings of key terms in that document. We understand that many readers will find some or all of these terms commonplace and not requiring special attention. For that reason, it may be best to skim the list and either skip the section entirely, read selectively, or read all the section—depending on your own sense. Consider also coming back to specific terms when they arise and are discussed in the main body of the document.

Understanding

To the authors' credit, the CCSS-M addressed directly, rather than avoided the issue of "understanding" in mathematics. All teachers appeal to that term at times, but we are often not so clear about what we mean when we say that a child does or does not understand a mathematical idea. The authors link the meaning of the term to students' capacity to justify why particular mathematical relationships are true.

> These Standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from. (p. 4)

So justification involves two essential elements: Students' ability (1) to explain in their words why they think particular relationships are true generally (not only in specific cases), and (2) to base their explanations in other mathematical principles or relationships that have previously been justified. Justification means appealing to mathematical ideas that other students in the class consider true, for good mathematical reasons.

What does this view of understanding mean for measurement in particular? Measurement can be understood as a collection of procedures that produce measures, like, "9 paper clips long," "3 inches," and "16 square centimeters." But what guarantees that these procedures are correct ways to measure, whether they are "standard" procedures or "invented" methods? Justification of all measurement procedures rests involves the appeal to basic principles of identical units, the iteration/tiling of units in the space to be measured, and ways we keep track of how many units we used an object or space. So understanding measurement procedures means that we can show how what we have done fulfills those basic principles. In contrast, claims that "we just do it that way" do not yet indicate understanding.

Discrete vs. Continuous Quantity

Unlike the term, "understanding," that appears in many contexts in the CCSSM, the terms "discrete quantity" or "collection" and "continuous" or "measurable quantity" do not appear as often. Yet thinking about the measurement component of the CCSS-M in relation to the *Number and Operation* domain in the elementary grades requires attention to and understanding of this distinction. It is basic to much of mathematics up through calculus (and beyond). Discrete quantities are collections of individual objects, things we can count. In most curricula and in the CCSS-M, discrete quantities are the basis for understanding what whole numbers are and what arithmetic on those numbers means. But there are also many quantities that we cannot count; these are the quantities that we must measure. Again, think about "quantity" here in terms of a measurable attribute of an object. So length, area, volume, weight, time, density are all continuous quantities. We make them discrete by cutting them up into equal-sized units so that we can count those units, but they don't start that way. For many reasons, thinking of the world only in terms of discrete quantity can make it harder to learn measurement. In our discussion of the CCSSM's treatment of length, area, and volume in this document, we will come back frequently to this issue. Some of students' documented struggle with measurement can be explained by interpreting continuous quantities in discrete quantity terms.

Geometric measurement

Starting at Grade 3, the authors of the CCSSM use the term "geometric measurement" to describe some, but not all measurement competencies expected at that and subsequent grades. But the authors do not explicitly state what they mean by the term, though researchers (**Battista, reference; others**) have used the term to refer to the measurement of space (length, area, and volume) and angle. Since the CCSSM authors do not use "geometric measurement" in their discussions of length measurement, their meaning may not be identical to Battista's. At Grade 3, area measurement is introduced and the relationship between area and perimeter of polygons is discussed. So one possible reason for introducing the term at that grade (and not before) is that in two- and three-dimensional space, students must think more carefully about what attribute of the shapes they are measuring and how these attributes are effected by the geometric properties of those shapes. Since one Grade 4 focus is angle and angular measure as "geometric measurement," it is clear that the authors include (at least) angle measure and measurement in two and three dimensions as important parts of "geometric measurement." But to understand for sure what the authors mean by this term and why they have appear to have excluded length measurement from it, clarification from the authors would be needed.

Measurable attribute

Starting in Grade K, the Standards talk about the measurable attributes of objects, with length as a frequent example (e.g., p. 10 and 12). But physical objects in our world have many attributes, including color, texture, hardness, as well as length and weight. Most physical attributes of objects are measurable, even if some (like texture) require complicated techniques. Length is an important initial measurable attribute because it is accessible to young children.

But because all objects have many attributes, we cannot assume that all children will immediately focus on one attribute of a given object, or even one of its spatial attributes. For example, to say that one object is "bigger" than another usually does not yet make clear what spatial attribute the speaker is thinking about. Being clear to others about specific measurable attributes of objects means using language carefully. Because children may not appreciate the possibility of confusion, teachers must be leaders, asking for clarification from students when their descriptions are unclear and helping the class use language that helps everyone understand which attribute is being described and measured. Two key ideas that help with this work are (1) remembering that all objects have many attributes and multiple spatial attributes and (2) watching for words like "bigger" and "smaller" that are often ambiguous on the specific attribute and so in need of further clarification from the speaker.

Unit

The concept of unit lies at the heart of all measurement, including spatial measurement. All measurements are counts of units, e.g., "3 inches" and "4 cubic meters," so we cannot produce a measurement without first selecting a unit. Then we use that unit to see how many we need to fill up or exhaust the space we want to measure, whether it is empty space (like the distance across the classroom0 or the length of an object.

For most measurable attributes, there are many units. For example, we have two different sets of "standard units" of length (English or customary units like inch and foot and metric units like centimeter and meter) and many "non-standard units" (like paperclips and toothpicks) that we use to measure length in classrooms. Unless we are careful, however, we can forget that the same is true for area and volume and focus only on square units for area and cubic units for volume. We should not forget that sheets of paper, index cards, rectangular tiles, and beans are appropriate units of area and often very useful, depending on the surface whose area we want to measure. Similarly, Styrofoam "popcorn" is a non-standard unit of volume widely used to fill empty space in packing, as are non-cubic boxes.

Because they are physical objects, non-standard units like a traced and cut-out paper "foot," paperclip, and paper rectangle or square have multiple spatial attributes, including a length, a width, a distance around, and a surface whose area we can measure. So it is important to decide, when we use and then talk about non-standard units, which attribute of the object we are using as a unit. When we select such an attribute, we need to make sure that the attribute of the unit matches the attribute of the object or space we want to measure. So for example, to be clear to others, we need to know (and usually say) that we will use the length (the long dimension) of the paperclip to measure the width of a sheet of paper. And when we use the term "non-standard unit" we need to be careful to remember that we are selecting objects for their spatial attributes and using one of those attributes as our unit. An object like a paper square or rectangle can be used to measure both length and area, depending on the attribute (and therefore unit) we want to use.

Iteration/Iterating

The idea of iteration is already present in the idea of unit. If we did not use a unit repeatedly in our measurement process, it would not really be a measurement unit. In spatial measurement, iteration is the process of repeated application of the spatial unit until we have filled up or exhausted the space we are measuring. To exhaust the space accurately we need to be careful about our placement of units; we cannot leave gaps between units, or overlap units, or fail to note and account for where units fall outside of the attribute we are measuring.

Though the idea of iteration means the repeated application of a unit, in practice iteration can take two forms. If we have a sufficiently large collection of a non-standard units, say a large box of paperclips, we can use as many paperclips as we need to measure the attribute of the object we are thinking about, for example, the width of a sheet of paper. This process of applying many instances of the same unit is often called tiling. The term suggests area, e.g., "tiling" a floor in rectangular or square tiles, but it has a sensible and parallel meaning for length and volume as well. At times we don't have a sufficient collection of a non-standard units so we have to reuse the same object (as a unit) multiple times by moving it to new location adjacent to its last location. The movement of a unit to exhaust the space is the common meaning of iterating. Understanding iteration means seeing that tiling and iterating are two different images of the same idea—filling space with identical units. Here "seeing" means the ability both to visualize and physically demonstrating how iteration produces a tiling.

Composition/Decomposition & Composite Units

In many passages in the CCSSM, the authors discuss methods for measuring length, area, volume, and angle using the terms "decomposition" and sometimes both "composition" and "decomposition." Some readers may be more familiar with the term "partitioning," whether into parts of equal size or not. Decomposition into equal sized parts is equivalent to constructing and iterating a unit (that leaves no left-over space). Neither term, "composition" or "decomposition," is particularly complicated; these processes are essential to see how we can use geometric knowledge to help with measurement work. Simply put, "decomposition" means the breaking up some space (length, area, or volume) or angle into a number of pieces, without throwing any space or angle away. We often need to decompose when the space or angle we are working with is too big or complicated to handle. For example, imagine a two-dimensional figure that is composed of a square with a triangle "pasted" on one side and a semi-circle onto another. The area of that complex shape is much easier to determine if we "decompose" the complex shape into three parts and work on the area of each: (1) the square, (2) the triangle, and (3) the semi-circle. If we "decompose" a shape or angle, we need to "recompose" the measures of the parts to determine the measure of the initial shape or angle. It makes sense to decompose and recompose because all measures (length, area, volume, angle) are additive: Dividing up into parts (decomposition) and adding parts back together always "conserves" the length, area, volume, or angle of the original whole shape (or angle). Decomposition is always possible (as long as you recompose), but some decompositions (and not others) are more helpful in simplifying a difficult measurement problem.

For area and volume measurement in particular, making/seeing and using "composite units" is specific kind of decomposition and composition. For area, rows and columns of squares are composite units because they contain single units (single squares) and we can count them as single units, e.g., four rows of eight squares each. Seeing rows and columns as repeated composite units is essential for understanding how multiplying of lengths can produce the area of a rectangle. Similarly, there are many ways to make/see and use composite units in three-dimensional prisms to help determine their volume of prisms. If we imagine any such "box" (a three-dimensional array of cubes), the bottom layer is one composite unit. We can iterate that unit through the height of the box to determine the volume. Alternatively, the cubes that form any face or slice of array are also composite units and they can be iterated through the third dimension to determine that volume in a different way. Seeing composite units and then moving/iterating them usually provides quicker and smarter methods of measurement. This is an important idea for teachers because students don't always "see" such composite units without assistance. If they don't see your composite unit, they cannot work with it.

"Composite units" provide an important link between measurement and number and operation (**Langrall et al., reference**). Units of 10, 100, and 1000 are also composite units, each of which combines 10 smaller units of the previous. So, there are 10 units of 10 in 100, and we can iterate that unit of 10, 10 times, to make the whole of 100. So the structure of the base-10 number system is grounded on essentially the same idea as forming composite units and iterating them to use the space in a shape or angle. The difference is with measurement that our composite units are not always powers of 10; we make and use them to fit the space (shape or angle) at hand.

Section III: The Mathematical Practices As They Relate to Measurement

It is clear that the eight mathematical practices are central to CCSS-M's vision of all students learning the powerful mathematics grade by grade. On page 8 in particular, the practices are seen as key elements in helping students learn core content (especially mathematical procedures) with understanding. However, the descriptions of the practices are quite brief (one paragraph per practice) and are not well-illustrated across content areas and grades (or grade bands). In fact, the authors express the hope that curriculum developers, designers of assessments, and programs of professional development will now take up the task of connecting the practices to core content in context of classroom teaching. In recent public presentations, the authors have also suggested that subsequent documents will illustrate the meaning of the practices more widely (across content) and clearly than the present CCSSM document does.

Whether those documents arrive in your school (or not) and whether they prove useful in understanding the meaning of the practices (or not), the practices are worthy of focused professional discussion with your colleagues about what they might mean across grade levels and content areas. Others, including professional organizations like the National Council of Supervisors of Mathematics, may enter this discussion with useful guidance and interpretation, but specific direction from others is no substitute for the collegial work of teachers, working together in their own schools and grade bands. Understanding the practices does not mean waiting for more clarity from the authors. As they acknowledge (p. 6), the current process standards of the PSSM and strands of mathematical proficiency in *Adding it Up* are similar in spirit and good places to start your inquiry and discussion.

As you discuss the practices, be careful to avoid the possible faulty assumption that any chunk of work you do with students in your classroom need be an example of <u>one and only one</u> of the practices. It is likely that good work by teachers supports the development of multiple practices at the same time—though it may not support all of them simultaneously.

Our discussion here focuses on the specific content area of measurement. The intent is to provide a way of linking the CCSSM authors' presentation of each practice to specific issues in understanding and doing measurement. The interpretive comments below will clearly make more sense if you read the short description of each practice in the CCSSM first.

<u>1. Make sense of problems and persevere in solving them.</u>
Some aspects of this practice apply easily to measurement and some do not. From one perspective, measurement only begins if there is a meaningful problem to solve, e.g., "how long is this object?;" "how far around is this table?;" "how much concrete do I need for this section of sidewalk?" In that sense, measurers often already have a problem and a clear plan for solving it. But deciding what attribute to measure, which unit of measure is best, and how to deal with complex shapes and fractional units are all "problematic" aspects of measurement that must be raised and resolved for measurement work to be meaningful and successful (not a rote procedure). One clear lesson of this practice is that teachers need to help their students agree on the measurement problem to be solved and make sure that different students are not thinking of different problems suggested by the same situation. For example, different students may focus on different attributes of the same object (like a rug in the classroom) but use the same words to

describe that attribute or what they intend to do to measure it. A second set of aspects of this practice relate to reasoning and communicating—both central processes in the *Principles and Standards for School Mathematics* (PSSM) (NCTM, 2000*)*. The authors state that proficient students will typically check their result with another method, ask themselves if the results make sense, and work to understand the approaches of their peers. All these sub-practices suggest that proficient students should aim to do more than produce an answer (in this case, a measurement). Teachers can clearly play a central role in making their classroom a place for students find these sub-practices sensible and highly valued.

2. Reason abstractly and quantitatively.

In the elementary grades at least, doing measurement thoughtfully means reasoning quantitatively, as the terms "quantity" and "attribute" (for measurement) are very close in meaning. Later, as soon as symbols are introduced to stand for the attributes of figures, e.g., "L" for the length of a rectangle, proficient measurement thinking involves both aspects highlighted in this practice: Students' reasoning must be quantitative in that they can easily generate many specific examples of rectangles with different lengths (L) and (for example) describe how the perimeter (P) is effected by changes in the length. But their reasoning must also become abstract, at least in time, in order to carry out the same sort of reasoning by thinking about and talking about how changes in L effect changes in P (or area) directly, without appeal to specific examples.

3. Construct viable arguments and critique the reasoning of others.

This practice, more than any other, calls attention to the importance of communicating our mathematical reasoning. It expresses, in capsule form, the same content as *PSSM's* (NCTM, 2000) Communication "process" standard. Understanding some mathematical idea is unlikely if we cannot effectively express that idea in ways that others can understand and appreciate. Where this practice applies generally to all areas in mathematics (including measurement), what makes arguments viable (effective) in measurement and what aspects of measurement reasoning are likely to be violated and become objects of critique by students can be specific to measurement. Some elements that are common to measurement arguments/reasoning are: (1) the attribute of the object or shape one is reasoning and talking about, (2) which spatial measure matches that attribute, (3) the units chosen for use, and (4) the number of units required to tile that attribute along the method used to determine that number (e.g., applying a ruler is different from counting). The ability to present viable measurement arguments and evaluate the arguments of others appropriately will typically require attention to these issues. More generally, it is quite unlikely that students will become adept at expressing their measurement reasoning well and become skilled (and considerate) in their evaluation of their peers' reasoning if their teachers do not create a climate where those actions are supported and become increasingly "normal." Effective mathematical communication requires development and support; it will not be supported by a focus on silent, individual work.

4. Model with mathematics.

The focus of this practice is to use mathematics well by attending both to features of the problem situations "in the world" that generate the need for problem solving and the mathematics (the model) used to carry out those solutions or reasoning. The central message is that powerful mathematics is mathematics that gets used in the world. As the authors emphasize, modeling with mathematics involves making assumptions about situations and choosing appropriate mathematical ideas and tools to build models of those situations. When we focus on measurement, in contrast to say number or algebra, we can see that measurement is already a model of some piece of the world. When we measure—that is, when we assert that some object or shape has some measure, we are already modeling. We are choosing an attribute of the object or space to measure and an appropriate unit, applying a measurement process to that attribute, and producing an "answer" (a measure of the attribute) that carries some error (see Practice #6 below). Where it is still important to focus students' attention on the fact that the measure is not the attribute, measurement seems less in danger of becoming separated from the world than other topics in elementary mathematics.

5. Use appropriate tools strategically.

For the most part, the authors consider "tools" in the usual way that educators do: They are physical objects, things that we can pick up and hold, that aid or extend our thinking in important ways. If we understand what tools are doing for us, they can be very useful because they free up mental resources so we can think about other aspects of the problem. For example, if we are working on measuring the perimeter of a complex shape whose boundary is made up of both line segments and curves, then measuring the length of the line segments with a ruler allows us to focus on the more challenging issue: What to do with the curved parts of the boundary. But since the authors include "estimation" as a tool for judging the reasonableness of answers and detecting possible errors, they invite us to consider other, non-physical tools as well. In measurement, it seems appropriate and useful to consider the computational formulas for finding the measure of attributes (e.g., perimeters, areas, and volume) as "tools." Just as our students can understand (or not) what the marks on a ruler mean, they can understand how formulas work as tools (or not). They should be able to explain how multiplying two lengths can produce a count of squares that cover a rectangle and hence produce a measure of its area. Or, as is often the case, they could see this transformation as a complete mystery. Teaching students to use measurement tools strategically clearly includes helping them see what physical tools like rulers and protractors do (how they work). But it also includes helping them see how intellectual tools (like formulas) work to help us measurement more quickly and efficiently.

6. Attend to precision.

Thinking about precision, estimating it, and matching the level of precision to the demands of the situation all are central issues in measurement, as all measurements include some error. In theory, any given line segment has a single exact length, once we choose a unit of length. In practice, any human attempt to measure that segment will produce an approximation of that theoretical "true" length. In the primary grades, making precision an object of discussion is often productive. A length measure of "about 4 inches" may be fine in one context, where knowing how much more or less than 4 inches may be important in others. In the middle grades, it is important to consider how arithmetic can increase prior measurement error, especially when we are multiplying measures. Where the focus on precision is often to increase it (by decreasing error), making error the focus of study can lead from measurement into important ideas of statistics, especially the notion of distribution (see Lehrer & Kim, 2009).

7. Look for and make use of structure.

The short paragraph describing this practice uses the terms “pattern or structure” without defining either term. However, the authors supply numerous examples of numerical and algebraic structures as well as some geometric ones. Consistent with these examples is the following meaning for “structure” or “pattern”: A mathematical relationship or concept, visible in some representation (including algebraic, numerical, graphical, and geometric), that appears across many contexts. So for example, in the application of the distributive property to the trinomial, “$x^2 + 9x + 14$,” the structure is not the fact that 9 = 2 +7 and 14 = 2 x 7 in this case alone, but the more general pattern that the middle term will always be the sum of the factors that compose the final numerical term—as long as the trinomial is factorable with integer terms.

With this meaning of structure in mind, are there measurement structures and if so, what are they like? Generally speaking, structure is present (and important) in all mathematics domains; the trick to know what one is looking for, as the nature of structures and how they are represented can vary across different domains. One important kind of measurement structure is revealed in the general formulas for area of polygons/circles and volume of prisms and cylinders. The common structure of area formulas is that they must involve the multiplication of two lengths. The nature of and relationship between those two lengths differ across geometric objects; the formula for the area of a rectangle requires the length of the shape and the width perpendicular to it, where the formula for the area of a circle multiples the radius by itself. But common to both is the fact that area requires that multiplicative composition that creates units of area measure. Someone who understands that structure knows to look for it and rule out any proposed area formula or rule that lacks that mathematical relationship. A similar structure arises for volume, with three lengths. Moreover, the fact that the area of the base multiplied by the height produces the volume of any prism (rectangular or not) and any cylinder, despite the evident differences in their three-dimensional shapes, is another important example of measurement structure.

Though all students enter school with abundant curiosity, we should not assume that the disposition to search for and express structure arises naturally in everyone. But with appropriate modeling and support, most all children are capable of doing so and become better learners of mathematics when they do. Teachers need to notice and call attention to common structures across mathematical domains, including measurement, and regularly ask students to search for them and support them in their efforts to express them in their own words.

8. Look for and express regularity in repeated reasoning.

This practice calls for attention to common (“repeated”) steps in problem solving and for students to propose/conjecture general statements that capture that regularity. This search for and expression of general patterns is fundamental to mathematical work. As was the case for Practice 7, the authors’ examples focus on algebraic content and expression. In the domain of measurement, however, there is at least one broad, powerful, and quite general regularity that current curricula, state standards, and the CCSSM itself all overlook. That is the common conceptual core that underlies all spatial measurement and indeed, all measurement more generally. It is remarkable that the core notions of (1) identical units, (2) unit iteration, (3) the inverse relation between unit size and resulting measure numbers, (4) the additive nature of like measures, (5) the notion of zero, and for area and volume and many other quantities, (6) the multiplicative composition from other quantities, like lengths, are introduced and discussed for length (separately), for area (separately), or volume (separately)—if they are discussed at all! But this conceptual core generally holds true across measures. (Multiplicative composition is not applicable to length, time, and weight). The lack of attention in curricula and standards to this conceptual “regularity” suggests to students that each new measure they encounter is new mathematics, when that is largely not true. It is important to add that attention to kind of conceptual regularity(a common conceptual core) can be supported in classrooms in ways that directly engage students. For example, when the exploration of area measure begins, the simple suggestion, “let’s think back to what we found for length,” can be a powerful entry into the joint (teachers and students) search for common relationships that bridge specific measures. Of course, that approach can only succeed when prior work with length has raised and explored the core concepts listed above.

Section IV: The Treatment of Length Measurement in the CCSSM

Overview

The authors of the CCSSM follow the practice of current state standards and most curricular approaches in introducing students to measurement generally through the measurement of length. They begin with the qualitative comparison of length and other attributes of objects (what is more and what is less?) in Grade K, e.g., the heights of two children. Measurement proper—determining length as some number of length units—begins in Grade 1. A careful reading of the document suggests that intended "units" at this grade are "non-standard," though that term is not used nor are examples of such "units" given at this grade. The iteration of units is presented as tiling, rather than the movement of one unit across a space or attribute. The important concept of unit-measure compensation (that smaller units of length produce greater measures) is explicitly mentioned, at both Grade 1 and 2. Measurement with standard units, the use of rulers, and the estimation of length are highlighted at Grade 2. Also at Grade 2, whole numbers are to be represented as lengths from zero on the number line.

Perimeter is the focus at Grade 3, when area measurement is introduced and emphasized. Explicit attention is given to understanding the difference between perimeter (a linear measure) and area for plane figures, but the discussion does not help teachers understand why students often confuse the two measures or provide guidance on how to help them understand the difference. The Grade 4 emphasis areas are unit conversion and applying the perimeter formula for rectangles. Unit conversion starts with translating from smaller to larger units and then moves, in Grade 5, to conversion from larger to smaller. The measurement of the circumference of circles is mentioned at Grade 7 (though almost in passing); the Grade 8 focus is on the Pythagorean Theorem, where length has become "distance."

In addition to the area/perimeter distinction, the document does not address or explain a number of well-known challenges that students face in achieving these goals. We highlight four.

• There is no discussion of how students can use units in ways that fail to the fill the space to be measured. Understanding length "units" means knowing how and why to use them; knowing how and why does not necessarily follow from putting units in the hands of children. We discuss some of patterns of faulty unit use/placement; the rationale for these ways of thinking; and what instructional moves can help to address them.

• CCSS-M does not identify, distinguish, or explain the customary and metric systems for length measurement at Grade 2, but intermixes customary units and metric units in Grade 2 standards. One missing issue here is the rationale for standard measurement systems (in contrast to personally-meaningful units, e.g., "my foot" as a length unit). Another is the origin and pattern of use of customary and metric units for length measurement.

• No standard clearly connects the tiling or iteration of units to the structure of marks on rulers, nor is the importance of zero (and scale) explicitly mentioned. As was the case with "units," these absences suggest, quite incorrectly, that rulers are easy tools for students to pick up and use appropriately. In fact, many students see the marks on rulers as objects to count (that is, they are not thinking about intervals of length and continuous quantity.

• There is no acknowledgement of the potentially confusing language we use to talk about length. Most everyday objects and two-dimensional shapes have multiple attributes that are lengths. For example, the classroom door has at least three attributes that are lengths, the "height," the "width," and the "thickness," but none is the "length." For rectangular objects, the "length" is one of length (e.g., the long dimension of classroom board), but so is the "width." We usually consider the spatial orientation of rectangles when we determine their "base" and height," but those terms may seem "wrong" when we rotate that rectangle 90 degrees and its "base" is vertical. Our point is not to critique our language for this spatial measure (it is the language that we have) but to note our concern about the silence of document on this crucial issue.

In our discussion, we have intentionally focused on length measurement content in Grades K through 4 because the bulk of length content appears in those years and, if we are successful, that is when most students will come to understand length as a quantity and how to measurement it. Our discussion of length content in Grades 5 through 8 is briefer.

Length Measurement Standards, Grade-by-Grade

Unless otherwise noted, all standards listed below come from the *Measurement and Data* domain. Content quoted from the CCSSM is given in italics; our interpretive statements are in plain text. The term "introduction" refers to the small number of "critical areas" and the brief standard statements that appear at the beginning of each grade-specific section of the document.

<u>Kindergarten (length)</u>

The introduction to Grade K standards (p. 9) includes a focus on geometric ideas but does not explicitly mention measurement. Two Grade K standards in the *Measurement and Data* domain address measurement (p. 12); one standard in the *Geometry* domain does so (p. 12).

• *Describe and compare measurable attributes.*

> *1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.*
>
> *2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.*

• *Analyze, compare, create, and compose shapes [Geometry].*

> *4. Analyze and compare a variety of two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, component parts (e.g., number of sides and vertices) and other attributes (e.g., having sides of equal length).*

These standards focus on qualitative comparison: Which is more or greater, not how much more or greater. Three methods are commonly used by adults and young children for making these comparisons. One is <u>direct comparison</u> where two objects are placing side by side on a level (or flat) surface and the greater height (or length) is noted. Another is to examine two objects <u>visually</u> and select the one that appears longer/taller. A third is <u>indirect comparison</u> where a third "intermediate" object that is longer/taller than first object is directly compared to the second. (Research suggests that Kindergarten students more quickly appreciate and apply the first two strategies (**Barrett & Clements references**). Perhaps for this reason, the CCSSM authors wait until Grade 1 to discuss indirect comparison explicitly.) Qualitative comparison is important because it provides the motivation for numerical measurement. Simply knowing that one object is longer/taller than another is sometimes sufficient but often leads to a related question, "how much longer/taller?"

In these standards, it would be easy to lose sight of the important idea that objects have multiple spatial attributes that are lengths—not one. Standard 1 includes this possibility without stating it explicitly. Children's heights may be among the most, if not the most important and visible aspect of their bodies, and the length of pencils may appear one-dimensional. But most physical objects in the rest of the wider world (and the world of classroom) are multi-dimensional. For example, the classroom door has height, width, and depth/thickness, all of which are lengths. Spending time clarifying which attribute of two objects can be compared is wise so that all in the classroom are thinking about the same comparison. Similarly, in the *Geometry* standard, which shape is "big/bigger" or "small/smaller" may need further discussion and clarification since children may refer to either parts of the shape (the lengths of polygon sides) or the whole shape. The issue of the meaning of descriptive terms for length anticipates numerous specific issues of ambiguity in the terms we use to describe and measure shapes (see below).

Second, the document does not clarify (or identify as an issue) what criteria separate measurable from non-measurable attributes. Young children may accept that length and weight are measurable but what about color and shape? We see these attributes as aspects we can "classify" but not easily measure, but many Grade K children will need support to appreciate this distinction.

Grade 1 (length)

The introduction to Grade 1 standards (p. 13) identifies measurement as one of four "critical areas" at this grade: *Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and the transitivity principle for indirect measurement. [Footnote: Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.]* As the specific standards make clear, "measurement" in Grade 1 seems to mean "length measurement."

• *Measure lengths indirectly and by iterating length units.*

1. Order three objects by length; compare the length of two objects indirectly by using a third object.

2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.

Standard 1 builds on the pairwise focus of qualitative comparison in Grade K. If children can order three objects by their lengths, it is likely that they can also put larger numbers of objects in order, one new object at a time. But the ability to systematically put in length order a jumbled collection of objects depends on seeing the (one) length attribute of each object.

Standard 2 introduces some of the most central ideas of measurement (generally) via length measurement—the notion of unit, tiling, exhaustion of the quantity to be measured (in this case, one-dimensional space), and the enumeration of units. (Note that the notion of iterating a single unit [in relation to "tiling"] is not introduced in Grade 1—but is featured at Grade 2.) Where each of these ideas is central, children do not immediately appreciate their importance. Where some Grade 1 children may, considerable research shows that (1) these ideas are not obvious to all children, and (2) they fail to satisfy these ideas in well-documented ways. Teachers can look for these flaws approaches in students' measurement work, as long as they are initially free to use length units in ways that make sense to them. If you model the length measurement process for them (Grade 1, with non-standard units), you cannot be sure that they understand why your approach to unit placement is needed and correct. Indeed, differences among students' methods of placing units (both "right" and "wrong") are excellent contexts for making these core ideas visible as issues to see, discuss, and understand. There is no discussion of rulers as length measurement tools at Grade 1; standard units and rulers are introduced in Grade 2.

If we follow the authors' suggestion to limit length measurement to objects whose length will be a whole number of the given unit, we can anticipate how some children will use non-standard units. For each "misconception," we highlight connections to core ideas we presented and discussed in the opening sections of this document (see Section II). First, if two collections of non-standard units are available for use, some children will be content to fill up the space to be measured (e.g., the linear space along side of the object) with some units of each type. Some will not be bothered by a mixture of longer and shorter units in their measurement; they are working to fill up (exhaust) the space. Such measurements, e.g.,"3 tiles and 4 cubes," does make sense, as it accurately describes an attribute of the target object, but it violates common practice in measurement. We name measures as some collection of <u>one</u> unit not many, primarily because that practice makes operations on and with measures (especially comparisons and arithmetic operations) possible and meaningful.

Second, some children will be satisfied that they have measured a length attribute if they have correctly aligned one unit with one endpoint of that attribute and another with the other endpoint, even if they have left space between their placement of other units in-between (NRC, 2008). This error is more likely when the length attribute is <u>not</u> a whole number of the given unit (because children are not sure how to deal with additional space that calls for a subunit), but also occurs when some number of whole units will exhaust the space. It comes from children's greater attention to the endpoints of paths and objects than to the space between. Third, we cannot always assume that children will understand how to place units sequentially along a path parallel to the attribute to be measured; they may lay their units on a line at an angle to the attribute, perhaps because the length attribute itself is not clear to them.

If children understand length measurement and can appreciate, if not clearly state the ideas listed in Standard 2, many will naturally want to measure objects in the classroom whose lengths will not urn out to be some whole number of units. Since these situations lead naturally to the need for (fractional) subunits, this interest should not be discouraged, rather anticipated. Early efforts to express fractions of units as "a bit more" or a "half" are sensible and lay the basis for further work with smaller subunits, either non-standard or standard.

<u>Grade 2 (length)</u>

The introduction to Grade 2 (p. 17) continues and extends the Grade 1 focus on length measurement as one of four critical areas: *Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.* The number of specific standards that directly concern length measurement at Grade 2 is even greater than in Grade 1.

• *Measure and estimate lengths in standard units.*
  1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
  2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
  3. Estimate lengths using units of inches, feet, centimeters, and meters.
  4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

• *Relate addition and subtraction to length.*
  5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
  6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

• *Represent and interpret data***.**
  9. Generate measurement data by measuring whole-unit lengths of several objects, or by making repeated measurements of the same object. Show the measurements by making a dot plot, where the horizontal scale is marked off in whole-number units.

• *Reason with shapes and their attributes [Geometry]*
  1. Recognize and draw shapes having specific attributes, such as a given number of angles or a given number of equal faces. [Footnote: Sizes of lengths and angles are compared directly or visually, not compared by measuring. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

Six aspects of this list most merit discussion. First, the notion of the compensatory relationship between the size of a length unit and the resulting number of units required to measure an object is emphasized, both in the introduction and in Standard 2. Second, the focus at Grade 2 is on standard units of length—though that term is not used explicitly (Standard 3). Third, consistent with that focus, rulers are introduced as length measurement tools (Standard 1). Fourth, even though number lines are often used to indicate, compare, and operate on numbers representing discrete quantities (chips, centimeter cubes, etc.), explicit attention is to given to representing and operating on lengths (addition and subtraction) on number lines (Standards 5 and 6). Fifth, lengths are quantities that can be composed and decomposed additively (that is, by addition and subtraction) but not yet by multiplication and division. Sixth, the CCSSM departs from current curricula and teaching practice by providing a doorway into statistics, specifically using repeated length measurement to generate meaningful data to graph (Standard 9). We discuss each in turn.

Some fundamental concepts of measurement that can be grasped in the measurement of length are (1) that the same object can be correctly measured with different length units, (2) each unit produces a different length measurement, and (3) smaller units produce larger measures (because you need more of them to exhaust the space). The first element underlies and justifies the use of different systems of measurement (customary and metric) that each include larger and smaller units of length (subunits). The third element underlies the procedures for converting between smaller and larger length units: That smaller units of length measure necessarily produce larger length measure numbers. For example, measuring in centimeters will always produce a larger length measure (of some object) than if the same object is measured in inches, because one centimeter is smaller length unit an one inch. Work on this compensatory relationship does not in any way require standards units and can be profitably explored with non-standard units in earlier grades.

Where standard units appear to replace non-standard units in Grade 2, the authors provide no logic for that transition. Many reasons make standard units more useful than non-standard units, but some may be difficult for second graders to appreciate. Standard units of length are required for standard units of area and volume (e.g., square and cubic inches, respectively). Standard units increase the range of application of measurement systems when more people use the same units and systems of units. The history of the development of the foot as a standard unit of length in the customary system is helpful here: Originally, a "foot" varied with the size of the person doing the length measurement with his/her feet.

Even if children appreciate the need for standard units, they may or may not understand the marking of inches (or centimeters) on the rulers when those tools are first introduced. Rulers mark locations ("1" means the end of the first interval) that may either be seen as <u>collections of objects to count</u> or as the endpoints of successive equal intervals. Understanding what rulers show will require, for some students at least, explicit attention to how collections of length units (e.g., inch line tiles and centimeter cubes) are related to the marks on rulers. The evidence that our students do not understand rulers well is extensive and consistent (**NAEP references; Cullen dissertation; UC Baby Lab paper**). An important test of understanding is to ask children to measure objects with "broken rulers"—those with no zero mark, because success requires knowing the meaning of the available marks.

Similar considerations apply to number lines. They can serve as an important representation of (continuous) length, as well as (discrete) collections of objects. But for the same reason discussed above for rulers, the marks on number lines are ambiguous. They can either be interpreted and used as locations of discrete quantities (7 apples) or as distances from zero (7 inches). Considering the discrete vs. continuous quantity discussion in the opening sections of this document, number lines could serve as excellent sites for establishing parallels and considering differences between discrete and continuous quantities, in this case, length. But the CCSSM authors do not explicitly recognize this ambiguity.

Finally, the key insight in relating measurement to work with data is that repeated measurement produces a collection of different measurements, so the resulting data can be plotted on a two-dimensional graph. What is skipped over in that standard are the core statistical notions of variability and distribution. All measurements include some error, principally from the application of measurement tools (like rulers) by humans. Errors of different sizes across a set of measurements produces variability—a distribution of measures. Hence repeated length measurement is an excellent way to introduce all the fundamental statistical notions (range, measures of central tendency, variability/distribution, and measures of variability) in a meaningful way (Lehrer & Kim, 2009).

Two final comments. Note that the term "iteration" appears twice in the introduction but nowhere in the specific standards. Indeed the meaning of that term, if different from the notion of "tiling" introduced in Grade 1, remains unclear. We have distinguished the two terms early in this document because based on what we know from current research, the equivalence of "tiling" (filling up the space to be measured with identical units) and "iterating" (placing and moving one unit sequentially to accomplish the same purpose) is an intellectual achievement—not to be taken for granted. Elementary students may not easily see them as essentially "the same" measurement process, so work relating both tiling and iterating to the structure of rulers will be valuable. It will also lay important groundwork for understanding the computational formulas for area and volume measurement when they are introduces in later grades.

In contrast to geometric work at Grade K, work with geometric shapes at Grade 2 focuses on <u>specific</u> geometric features (number of angles and/or equal sides), not global/visual appearance. This is an important shift because it similarity among shapes will focus, at this grade and hereafter, on their specific geometric properties.

<u>Grade 3 (length)</u>
Both the introduction (p. 21) and list of standards (pp. 24–26) focus on area measurement at Grade 3. There is reference to length measure only in the fourth critical area that focuses on geometry: *Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify the shapes by their sides and angles, and connect these with definitions of shapes.* It is clear that classification of shapes "by their sides" means considering side length as a central feature of polygons.

• *Represent and interpret data.*

3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step —how many more and —how many less problems using information presented in scaled bar graphs. Include single-unit scales and multiple-unit scales; for example, each square in the bar graph might represent 1 pet, 5 pets, or 10 pets.

4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a dot plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

• *Geometric measurement: Recognize perimeter as an attribute of plane figures and distinguish between linear and area measure.*

8. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

• *Reason with shapes and their attributes.[Geometry]*

1. Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of *these subcategories.*

Overall, there is a modest amount of new content for length measurement at Grade 3. The focus of the "geometric measurement" standard is the perimeter of polygons. Competence with perimeter is expected both determining perimeter from information about side lengths and determining missing side lengths from information about the perimeter (and other side lengths). Grade 3 students are also expected to explore the complex (and interesting) relationship between perimeter and area, holding one measure constant and exploring possible values of the other. Standard 3 engages length measurement because graphing with units of the vertical axis (dependent variable) greater than one are suggested. This move engages the central concept of scale in measurement. Standard 4 calls for ruler measurement coordinating units and subunits, e.g., "9 and ½ inches" and the construction of corresponding scales on axes where these measures might be graphed. The single *Geometry* standard continues prior work in the analysis of two-dimensional shapes, now with a focus on specific geometric properties (angle size and number, side length and number).

With respect to perimeter (the measure of the boundary of a two-dimensional closed curve), the measure of polygons (or any path with corners) can introduce new challenges for students, depending on the tools available to them. If the sides of polygons are whole numbers of the standard units represented on the rulers students are using, finding the perimeter of polygons will likely be no more conceptually demanding than measuring any one side as a single segment. But if non-standard units are given for measurement or the polygon is drawn on a grid, some students will be drawn to place/count "corner" units that just touch the vertices of the polygons but do not contribute to the perimeter (**Battista references**). They seem to be linking finding the perimeter with spatially surrounding the given shape. If the sides of the polygon are not whole number of ruler units, then students must manage measurement in whole and fractional units.

It is unfortunate that measurement in units and subunits (half- and quarter-units) is stated in a "data" standard, as the shift from whole units only to units and subunits introduces the core notion of precision in measurement, starting with length. Fortunately, the ability to understand and construct "half" in a variety of contexts appears early in children's development (**Poithier & Sawada, others**) and "fourths" as "halves of halves" are understood quickly as well. But it is important to note that understanding "where" halves and fourths are on rulers requires more than understanding the relationship between part and whole. Children must understand (and be able to show how) subunits like halves and fourths are iterated on the ruler.

In contrast to work in prior grades on the classification of shapes based on visual analysis and physical comparison (e.g., placing one shape on top of another), children's reasoning is expected to take on a more logical character at Grade 3. On the one hand, now objects and figures with the same shape but different size, e.g., squares, are expected to be seen and grouped as a class of objects. But on the other, all four sided figures should be understood as quadrilaterals and some shapes (like squares) must be seen as a special subset (subclass) of rectangles. This focus on classification (naming a figure by its properties as well as its overall shape) presents problems and confusions for some children, who are thinking of naming as a one-to-one process. They see the statement that "a square is also a rectangle" is a violation of the basic process of naming things in the world. For these students, nested classification (squares, within rectangles, within quadrilaterals) will be difficult until they can appreciate other sorts of nested (hierarchical) systems of classification, such as kinds of flowers or trees, where students can be expected to know many examples of a single class. So "oaks" are trees, but there are different kinds, e.g. "red" and "white" oak, as well as many trees that are not oaks. "Chairs" is another class of everyday objects with clear subsets that are accessible to children.

Grade 4 (length)

The introduction to this grade (p. 27) focuses one of three critical areas on two-dimensional geometry, but it makes no explicit mention of measurement, of length or other quantities. In the specific standards, the focus is on unit conversion, for length as well as other measures. At Grade 4, conversion moves from larger units to smaller, so from smaller numbers of (larger) units to larger numbers of (smaller) units and therefore requires the operation of multiplication.

• *Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.*

> *1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36),…*
>
> *2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.*
>
> *3. Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

• *Represent and interpret data.*

> *4. Make a dot plot to display a data set of measurements in fractions of a unit (1/2, ¼, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in dot plots. For example, from a dot plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

Of the four standards, only the first (focusing on unit conversion) is new at this grade. Note the explicit direction that conversion in Grade 4 should move from larger units to smaller units, in this case for length, but also for other measured quantities; conversion from smaller to larger units is the focus at Grade 5. Though the authors do not state the connection explicitly, the justification of unit conversion lies in the compensatory relationship between the size of units and the number required to exhaust any space. Since smaller units (e.g., inches as compared to feet) occupy less space, more of them will be required to tile any given length. This makes sense when we think about measuring some object's attribute: We need more inches to measure across the classroom than we would if we used a larger length unit like feet. But it is also true for the units themselves: If we think about tiling the space of one yard, we clearly need more inches to do that than feet. And because tiling each larger unit requires the same number smaller units (we need 12 inches to "fill" each foot), it makes sense that the operation we use to convert the number of larger units into the equivalent number of smaller units is multiplication by the whole number ratio (in this case, 12 inches per foot). Because the compensatory relationship does not depend on the specific size of the length units, the logic also holds for metric units (centimeters versus meters) and for conversions between systems of length measurement(e.g., centimeters to inches).

The second standard focuses on the use of all four arithmetic operations to solve word problems involving a variety of measurable quantities, including "distances." Addition and subtraction of distances and lengths have been addressed in Grade 3 in the standard discussing problem solving with the perimeter of polygons. But the introduction of multiplication and division is new at this grade. Beyond unit conversion and multiplication (as discussed above), two other types of multiplication of lengths/distances are important to understand and distinguish. One type is usually called scalar multiplication: This is the mathematics of enlargement and shrinking. For example, " two and one-half times a given length" means putting two and one-half replicas of the length end-to-end and seeing what the total length is by adding or multiplying the given length by the scalar, "2.5." Multiplying a length by a scalar larger than one produces another length that is longer than the original; multiplying by a scalar less than one "shrinks" the original length to a shorter one. The second type is central to area and volume measurement; it is the multiplication of one length by another. This type of multiplication is often called multiplicative composition or the Cartesian product. The important idea here is this type of multiplication produces a quantity that is different than either factor. In the most familiar case, multiplying a length by a second length produces an area measure, not another length, as scalar multiplication does. Multiplicative composition is the "workhorse" type of multiplication in the sciences. In physics, "Force = mass times acceleration" is an excellent example; a "force" is neither a mass nor an acceleration. Likewise, "density," the ratio of mass and volume, is neither a mass nor a volume. Both are different quantities than the two they are "composed" from. In multiplication, new quantities are made from "old," and the units of new quantity bear the multiplicative stamp of the source quantities. We return to this issue in Section IV (below) on area measurement because of the central role that the multiplication of lengths plays in the determination of area measures for different geometric shapes.

Grade 5 (length)

The overview of Grade 5 standards (p. 33) focuses one of three critical areas on volume measurement; there is no explicit or implicit mention of length measurement. The Grade 5 standards extend the Grade 4 focus on unit conversion, now from smaller units to larger, thus decreasing numbers of units, and thus typically calling on the division operation.

• *Convert like measurement units within a given measurement system*

1. Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

• *Represent and interpret data.*

2. Make a line plot to display a data set of measurements in fractions of a unit (1/2, ¼, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. For example, given different measurements of liquid in identical beakers, find *the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

We expect that the conversion of length measurements from smaller to larger units will be more difficult for students to master than conversion from larger to smaller, and attention to the compensatory relationship first may help. Students should at least be able reason, with support, that the measure of a length in larger units should be smaller numerically than it was when measured in smaller units. We have include the data standard (#2) to highlight that plotting data requires the construction of axes and plotting points for measurements in fraction units requires the partitioning of whole units on the axes in a manner suitable to the data—at this grade into 2, 4, and 8 equal sub-units.

Grade 6 (length)

There is no explicit mention of length measurement in the introduction and only one reference in the specific standards. That single standard extends the Grade 4 and 5 focus on unit conversion. Conceptually, new content is extensive at this grade, where the central focus is on ratio, rate, and proportional reasoning. These relationships may involve length measures, e.g., speeds in terms of miles per hour. But the discussion of ratios and rates does not draw specific attention to length measurement. Attention is given to area and volume measurement in the introduction. Indeed an entire additional paragraph is devoted to these topics in the introduction, though it is not portrayed as a fifth "critical area" for this grade. The authors may be hesitant to depart from the pattern of either three or four critical areas per grade. Statistics makes its first appearance at Grade 6.

• Understand ratio concepts and use ratio reasoning to solve problems. [Ratios and Proportional Relationships]

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Grade 7 (length)

The introduction (p. 46) includes one explicit reference to length (finding the circumference of circles) in one critical area focusing on area and volume and anticipating work on congruence and similarity in subsequent grades. *Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in Grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections.*

• Analyze proportional relationships and use them to solve real-world and mathematical problems.[Rates & Proportional Relationships]

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks ½ mile in each ¼ hour, compute the unit rate as the complex fraction ½/1/4 miles per hour, equivalently 2 miles per hour.

• Draw, construct, and describe geometrical figures and describe the relationships between them. [Geometry]

1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

• Solve real-life and mathematical problems involving angle measure, area, surface area, and volume. [Geometry]

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Where there was no explicit reference to length measures as quantities that enter into ratio and rate relationships in Grade 6, measured quantities (including length and area) are explicitly named as examples in Grade 7. The focus on ratios extends to the geometry in the construction of scale drawings. The document calls for knowing the formulas for the area and circumference of circles but gives no guidance on how these formulas should be taught so that students know them. Absent any such guidance, the document allows for, if not suggests memorization. Though these relationships can be taught meaningfully, the role of π in each is difficult to account through any other means than approximation.

<u>Grade 8 (length)</u>

Length measurement receives explicit focus in introduction (p. 52) as distance, where one critical area focuses in congruence and similarity and the Pythagorean Theorem for right triangles. *Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons.*

• Understand and apply the Pythagorean Theorem. [Geometry]

6. Explain a proof of the Pythagorean Theorem and its converse.

7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Relations of congruence and similarity rest, in part, on judgments of corresponding lengths between shapes, equal lengths for congruence and proportionality between corresponding lengths for similarity. Understanding similarity beyond visual appearance ("same shape") therefore depends on the ability to apply ratio relationships to lengths of shapes. The authors appear to favor viewing the Pythagorean Theorem as a relationship between the areas of three squares (whose sides are the sides as the right triangle). But this famous relationship can be seen both as a relationship between areas and as a relationship between lengths. The authors appropriately want students to apply the Pythagorean Theorem to problems involving the distance between points and to side lengths in right triangles, but they do not tackle the difference between distance and length in geometry.

Grade 8 also references length measurement in the specific case of small measures that are best expressed in scientific notation. This standard does not seem to make new conceptual demands from the perspective of length measurement.

• *Work with radicals and integer exponents. [Expressions & Equations]*

4. Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

Section V: The Treatment of Area Measurement in the CCSSM

Overview

After work in geometry to explore and analyze the size and shapes of two-dimensional figures in the primary grades, work on area measurement begins in Grade 3. The document focuses on three methods of determining measures of area: (1) counting the square units that cover two-dimensional shapes, (2) computing the area of simple shapes (e.g., rectangles) by multiplying lengths, and (3) finding the area of more complex shapes via decomposition into simpler shapes and summing the areas of the resulting parts. Counting squares and computing the area of rectangles are the focus of Grades 3 and 4; work on more complex shapes follows in later grades. Two important deficits are: (1) the relatively weaker of discussion of the conceptual principles underlying area measurement (compared to length in Grades 1 and 2), and (2) the absence of dynamic, continuous, or motion-based representations of area. The current focus on static representations of area (e.g., rectangular arrays of square units) fails to support many students' understanding of how the multiplication of lengths makes square units and thereby area measures.

Comparing two-dimensional shapes of various sizes, composing and decomposing shapes, and partitioning geometric shapes into two or more equal parts, the activities mandated in the early grades, can support the development of area in later grades. Area measure is defined in Grade 3 as a discrete quantity (a count of unit squares) and substantial attention is given to area at that grade. The formula for the area of rectangles is introduced in Grade 3; its rationale lies with the fact that such multiplications produce the same number of squares as tiling the rectangle and counting the squares. No strong relationship is drawn to the row and column structure of that array of squares. Composing and decomposing into smaller rectangles to find area of rectilinear shapes without using new formulas is reinforced in Grades 4 and 5. Finding the area of other geometric shapes (right triangles, other triangles, special quadrilaterals, and polygons) is a focus in Grade 6. However, students are expected to derive formulas by composing/ decomposing and rearranging pieces using only the formula for rectangles. Contrary to current practice, the document does not emphasize the development of specific formulas for many shapes (e.g., those listed above). The area of circles is addressed in Grade 7, but no guidance is offered in dealing the difficulties of understanding π.

To their credit, the authors introduce a number of important conceptual principles that underlie and justify methods for determining area measures, many at Grade 3. The *meaning of a square unit* and its role in area measurement is highlighted, as is *unit iteration*, i.e., covering a figure with square units without gaps or overlaps. However, as was true for length, the focus is on <u>tiling</u> plane figures, not on iterating a single unit through the space. Area is defined as a "cover" of squares: Covering a region with N unit squares means its area is N square units. The *additive property* of area measure is also named and related, at least implicitly, to the process of determining the area of regions by decomposition. The *conservation of area under partitioning and motion* further justifies those procedures, but these fundamental principles are <u>not</u> mentioned. The area of shapes is compared and distinguished from their perimeter, a one-dimensional measure. However, the document does not help teachers understanding why students persistently confuse the area and the perimeter of the same shape.

In contrast, the methods of determining area and the justification for them receive less attention than is true for length measurement. The document skips over area measurement with units other than squares, is silent on some important conceptual properties, and fails to address well-known challenges for students' understanding of area, as distinct from length. We highlight five:

• The document presents the *structure of rectangular arrays* in identical rows and columns twice, once in a single Grade 2 standard on partitioning and once in the introduction to Grade 3. But it never states explicitly that the total number of squares in such arrays can be computed by multiplying the number of squares in a row (or column) by the number of rows (columns). Nor does it refer to rows and columns as *composite units*, though students' ability to "see" such composite units is important for their understanding of the spatial structure of rectangles (**Battista references**). Where there are three accessible methods for determining the area of rectangles (counting all squares, counting or multiplying with composite units, and multiplying length and width), the document discusses two and hints at the third. Excluding explicit discussion of the multiplication (or skip-counting) of composite units in fortunate, as teachers need to see and appreciate how that approach could support students' understanding of how "length times width" works less likely.

• Very little attention is given to determining area by *covering regions with non-standard units* (non-squares and squares whose sides are not standard units of length. Save one reference in Grade 3 to "improvised units," the idea of non-standard units is not present. The document also provides no motivation for the specific *geometric advantage of square units* over other units that can cover regions. This is a second way in which students' understanding of the formulas for area, beginning with rectangles, is weakly supported.

• The document does not state the *inverse relation* between the size of area units and resulting area measures, where it did so explicitly for length. Through Grade 8, no attention is given to the *conversion of area units*, which is difficult for students because the conversion relationship is no longer linear (1 square foot is not 12 square inches). By contrast, the document includes many references to the conversion of length units. In neither case (length nor area) does the document draw the conceptual connection between the inverse relationship and the logic of unit conversion (that conversion to smaller units <u>must</u> make larger measure numbers).

• Area measurement is cast as a discrete quantity throughout, that is, as a count of squares (whether by counting or by computation from lengths). There is no discussion of *continuous and dynamic (motion-based) representations* of area, such as "sweeping" a vertical line segment (or object) horizontally through space. Such representations hold promise for helping students understand how multiplication composes area from length.

• Students are expected to recognize perimeter and area as different attributes of two-dimensional shapes and distinguish between linear and area measures. However, no guidance for teachers is given in how to support that distinction or for understanding why the "confusion" of area and perimeter is so common.

Content that is relevant to area measurement is also discussed outside of the domains of *Measurement and Data* and *Geometry*. Representing the product of two numbers in rectangular arrays is expected to support students' understanding of operations (*Operations and Algebra Thinking*). Area measure is foundational for the development of fractions: Partitioning two-dimensional shapes into equal parts leads directly to the definition of fractions as a collection of those parts (*Number and Operation—Fractions*).

Area Measurement Standards, Grade-by-Grade

Unless otherwise noted, all standards listed from Grade K to 5 are given in the *Measurement and Data* domain and from Grades 6 to 8 in the *Geometry* domain. As above for length, content stated literally in the CCSSM is given in italics; our interpretive statements are in plain text. The term "introduction" refers to the small number of "critical areas" and the brief standard statements that are offered at the beginning of grade-specific section of the document.

Kindergarten (area)

In Kindergarten one of two critical areas is devoted to geometry: *Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.* There is no explicit mention of area measurement, in the introduction or the specific standards. The overall size of two-dimensional shapes (that is, the area or space enclosed) could be one of such property that students notice and describe, but the document does not indicate that expectation explicitly.

The two measurement standards focus on measurable attributes and qualitative comparison of objects, but do neither names the area or two-dimensional size as one such attribute. As we noted in Section IV, the focus in these standards is on length, and the document does not clarify what criteria distinguish measurable from non-measurable attributes.

• *Describe and compare measurable attributes.*

1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.

Three Geometry standards provide greater specificity on the focus on exploring and describing two-dimensional and three-dimensional shapes.

• *Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres). [Geometry]*

2. Correctly name shapes regardless of their orientations or overall size.

• *Analyze, compare, create, and compose shapes. [Geometry]*

4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

6. Compose simple shapes to form larger shapes. For example, "Can you join these two triangles with full sides touching to make a rectangle?"

These standards indicate that students should be able to identify the named shapes independent of their orientation in space. Though the document leaves the point implicit, the "analysis" of shapes in Grade K focuses on the geometric properties of vertices (corners), number of sides, and equal (or unequal) sides. Composing simple shapes to make another simple shape (Standard 6) is consistent with the introductory statement emphasizing identification, naming, and describing, but moves beyond it. Composition of simple shapes lays the experiential ground for later work on decomposition of complex shapes and for understanding the additive property of area measurement.

Grade 1 (area)

As we saw in Section IV, the measurement focus in Grade 1 is on length; there is no mention of area at this grade. The document devotes one of four critical areas again to geometry. *Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.* We see little difference between the geometric focus in Grades K and 1. The Grade 1 focus returns to plane and solid figures; appreciating that the shape remains unchanged when the figures is moved into different orientations; and composing shapes into other shapes. The introduction explicitly mentions "part-whole relationships" but does not clarify that quantity is continuous here not discrete (as it is in the *Number and Operations in Base Ten* domain).

Three specific standards address work with two-dimensional and three-dimensional shapes.

• *Reason with shapes and their attributes [Geometry]*

1. Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.

2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.

3. Partition circles and rectangles into two and four equal shares, describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

The first distinguishes defining attributes of shapes from others, but again does not address the more fundamental issue of which attributes are measurable. The focus on composition in Grade K is extended to more shapes, and partitioning into two and four parts is introduced. The focus on halving (to make two and four parts) is consistent with current curricular practice and with research that shows that making two equal parts (and successive halving) is an early cognitive achievement (**Pothier & Sawada, 1983, Confrey, other references**). The "understanding" that creating more equal parts within the same shape necessarily makes those parts smaller is fundamental for fractions and for area measurement (the inverse relation between unit size and number required to measure), but these connections are left implicit in the document.

Grade 2 (area)

The measurement focus at Grade 2 remains with length. The introduction continues the prior focus on composition and decomposition of two-dimensional shapes, devoting one of four critical areas to it, but without great clarity on what is new at this grade level. *Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.* We see little top-level difference here from the objectives stated for previous grades, save two exceptions: (1) area (and volume) are mentioned in the introduction but without further development in the specific standards and (2) the geometric focus of work shifts from seeing and interpreting shapes to making them. The goal of developing "a foundation for understanding area" is certainly worthy but, absent any discussion of conceptual principles for area, what it would mean to pursue that goal is left unclear.

• *Reason with shapes and their attributes. [Geometry]*

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

2. Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Standard 1 pursues the goal of constructing two- and three-dimensional shapes given their central geometric properties. Standard 3 extends prior partitioning work to circles, and importantly, to three parts—a difficult goal for children of this age for either rectangles or circles (**reference**). But Standard 2 is a central, if unmarked step in area measurement. Though no reference is made explicitly to "area," covering the rectangle in rows and columns of the same-size square is area measurement, if the square is seen as a unit of area and rows and columns are appreciated as composite units. But as stated above, no explicit attention is given to the importance of rows and columns for their space-filling and space-structuring roles, beginning with rectangles. Though "covering and counting" is a common early approach to area measurement in current curricula, this is the only reference to "covering" with squares prior to Grade 3, where the measurement focus turns to area.

<u>Grade 3 (area)</u>

Area measurement is a major content focus and the clear measurement focus at Grade 3. It is specifically named in the introduction (p. 21) as one of four critical areas: *Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.* Moreover, in two different locations in the introduction, area measurement is linked to whole number multiplication via rectangular arrays and area models. Given students' struggle to understand the standard formulas for computing areas (**references**), beginning with rectangles, explicit attention to the importance of structuring rectangular arrays in rows and columns is very positive. However, that idea is not developed (or even mentioned) in the specific standards that follow, and in the introduction, the authors immediately shift back to viewing rectangular arrays in terms of single units. The specific standards do not provide sufficient attention to the difficult and subtle shift from counting squares in a rectangular array (either counting all or counting via rows and columns) and multiplying lengths to determine the area of a rectangle. These three procedures yield the same result, but the document (and most current curricula) fail to link the processes (not just the outcomes) of counting to squares to the multiplication of lengths.

The fourth critical area in Grade 3 that focuses again on geometry also addresses area: *Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.* The new geometric focus here is the <u>classification</u> of shapes based on their definitions, not just—as in previous grades—on single properties. But more centrally to measurement, partitioning into parts of equal area is the foundation for representing fractions as numbers. Though we do not discuss that development in detail here, the *Number and Operations—Fractions* standards on page 24 state the point clearly. In sum, area is introduced in Grade 3 as measurable attribute of objects, and partitioning (creating equal units of area) provides the basis for developing fractions, in Grade 3 and in subsequent years.

Finally, the term "geometric measurement" appears without explanation in Grade 3. Inspection of Grade 3 and 4 standards that bear that label suggest that the authors include area, perimeter, and angle as objects of geometric measurement. This suggests that the authors use this term to indicate the measurement of properties of shapes in two dimensions (at least)—and to exclude the lengths of objects and segments. The term is also used in Grade 5 to frame the authors' discussion of volume measurement (see Section VI).

The first cluster of standards is sufficiently lengthy and detailed to merit discussion one standard at a time. To keep the top-level focus, we repeat the cluster label ("geometric measurement") for each of standards 5, 6, and 7.

• *Geometric measurement: understand concepts of area and relate area to multiplication and to addition.*

> *5. Recognize area as an attribute of plane figures and understand concepts of area measurement.*
>
> > *a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.*
> >
> > *b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.*

Area is introduced as an attribute of two-dimensional shapes; prior grades have focused on the geometry of these shapes. Standard 5a defines area units exclusively in terms of squares with sides of unit length. There is no explicit mention here of area measurement with nonstandard units—the first major difference with the document's treatment of length measurement (but see Standard 6 below). Where squares have a key geometric advantage in area measurement, chiefly the link between multiplying lengths and counting squares, other geometric shapes and everyday objects also cover effectively and appear commonly in the physical world. Perhaps more important, there is no discussion of the geometric properties of squares that make them special among area units. The lack of attention to other, non-square area units might lead students to conclude that area can only be measured in square units.

Standard 5b describes the process of tiling a region with squares. This statement is directly analogous to the tiling of length in Grade 1. What is missing from both discussions is important connection that one unit can be iterated through the space (for length or area) and produce the same count of units as a complete tiling of the same space. The difference between tiling and iterating (despite their equivalent results) is important because of the evidence that motion—physically moving the unit—may supply important experience and mental imagery in grasping the exhaustion of space (**UC Baby Lab study**). Here and through Grade 8, the document unfortunately draws no explicit connections between the conceptual elements common to both length and area measurement (and later, for volume). At best, it repeats them separately for each measure.

• *Geometric measurement: understand concepts of area and relate area to multiplication and to addition.*

6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

This is the first explicit discussion of different area units. All are squares whose side length is a standard length units in the metric or customary system—save the final element in the list "improvised units." We expect that this term is synonymous with "non-standard units" though the adjective "improvised" carries somewhat different connotations—improvised to fit a specific tasks context. Non-standard units include everyday objects such as beans (that do not easily tessellate) and sheets of paper or "stickee" notes (that do tessellate). As with "geometric measurement," the intended meaning of this term remains an educated guess.

It is important to note that the conception of area measurement expressed in Standards 5 and 6 is entirely discrete: Area is the count of squares with a side of some unit length that cover a two-dimensional shape without gaps or overlaps between the units. Nowhere in these statements is the view that area is amount of shape enclosed in such shapes, simple or irregular, or more generally, on both planar or non-planar surfaces. This continuous view of area seems important for supporting the authors' focus on composition and decomposition as methods to determine the area of complex shapes and the additive property of area measurement.

• *Geometric measurement: understand concepts of area and relate area to multiplication and to addition.*

7. Relate area to the operations of multiplication and addition.

a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

b. Multiply side lengths to find areas of rectangles with whole number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

*c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and b + c is the sum of a * b and a * c. Use area models to represent the distributive property in mathematical reasoning.*

d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into *non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.*

Standard 7 focuses on the application of the ability to determine the area of a rectangle to "real world problems" (parts b and d) and using the area of a subdivided rectangle to justify the distributive property (part c). We focus our discussion to two other major issues in the standard: (1) the treatment of tiling/counting squares and multiplying side lengths as two methods for determining the area of rectangles and (2) the additive character of area measurement. On the first issue, note that Standard 7 makes no reference to the structure of rectangles into equal rows and columns as was done in Grade 2 and explicitly in the introduction. Thus, in Grade 3, there is no explicit linkage between this structure and the multiplication of side lengths to determine area. Instead, Standard 7a simply points to the fact that tiling and counting produces the same count of squares as does the multiplication of lengths. The linkage between equal rows and columns as composite area units and multiplication of lengths could get lost.

Spatial structuring of rectangles into rows and columns—that is, the ability to "see" rows and columns as composite units—is mentioned as an important concept in area measurement in various research studies. Battista and Clements defined spatial structuring as "the mental operation of constructing an organization or form for an object or set of objects. (1998, p. 503). In spatial structuring objects are described by its spatial components. There is an identification of components, making up composites by those components and recognition of relationships between components and composites (Battista et. al, 1998). This structuring is indicated as an abstraction in the sense that "the mind selects, coordinates, unifies, and registers in working memory a set of mental items or actions that appear in the attention field." (Battista & Clements, 1998, p. 504). Sufficient evidence exists to show that the transition of understanding area as the count of squares (either by counting all in a tiling of squares or by counting by composite units where appropriate, such as in a rectangle) to understanding that area is the multiplicative composition of two perpendicular lengths is difficult for both teachers and students. Indeed it is a very subtle shift from one to the other. But the document does not serve the needs of teachers and students well by sidestepping, rather than addressing the issue.

Standard 7d states explicitly that area is an additive quantity: The sum (or difference) of two areas is another area. This fundamental property justifies the general procedure of decomposing a region into smaller regions, determining the area of each, and finding the area of the original region by adding the areas of the parts. Even more fundamentally, we can say that area is "conserved" (none is lost) when regions are decomposed into smaller regions. The authors of the document miss an important opportunity to state an important related property: Addition generates an "output" quantity that is the same as its "inputs;" multiplication generates a different "output" quantity than its "inputs" (Schwartz, 1989). This is an important key to unlocking the mystery of how the multiplication of lengths produces an area (a count of squares). Multiplication adds a new dimension to the shape whereas in addition the dimension remains the same though you increase the amount you have. Note that the development of multiplication as repeated addition (e.g., in *Number and Operations in Base Ten*) does not support students in understanding that the unit changes (linear to area) when we multiply lengths.

• *Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures*

8. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

So perimeter now appears as a property of two-dimensional shapes. Linear and area measures are distinguished, but the document does not address these issues sufficiently. Research indicates that students confuse perimeter and area (**references**), likely because (at least as a partial explanation) that all two-dimensional shapes have both measures: (1) a measure of the space enclosed/number of square units that tile the space, and (2) the distance around the boundary of the shape. The document neither explicitly states that perimeter is a particular kind of length measurement (e.g., the sum of the lengths of sides for a polygon) nor why it is (length measurement is additive). Nor does it provide any direct guidance about how teachers might help students to see both measures of two-dimensional shapes.

In addition to these *Measurement and Data* standards, area is mentioned in two other domains at Grade 3.

• *Represent and solve problems involving multiplication and division [Operations and Algebraic Thinking]*

3. Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

The standard suggests that arrays and "measurement quantities" (presumably areas) are appropriate for representing multiplication and division of single digit numbers. This is certainly a productive suggestion, but as the discussion above has indicated, how multiplication is working in three types of "models" is quite different. Counting all elements in an equal groups model, an array model, and an area model works equally well and raises no deep conceptual problems. The challenge arises when we stop counting single or composite units/elements and multiply lengths.

• *Reason with shapes and attributes [Geometry]*

2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.

Though we do not explore this issue in detail, in *Number and Operations—Fractions* the document develops fractions as numbers by partitioning a whole, taking some number of the resulting parts, and representing that quantity on the number line. But here in *Geometry*, we are reminded that the partitioning of the whole is indeed a partitioning of a region to create parts of equal area. This standard builds directly on Grade 2 *Geometry* standards discussed above. The inclusion of this *Geometry* is important because it is an explicit link between fractions, which are typically seen as a crucial topic in number and operations, but whose very meaning depends on area measurement and equal area units. The authors of the document leave this important connection across domains to the reader.

Grade 4 (area)

The introduction (p. 27) returns to the geometry of two-dimensional shapes in one of three critical areas, but makes no explicit mention of area measurement. The new geometric focus is on symmetry. *Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.* Outside of the *Geometry* domain, one standard focuses on the application of formulas for area and perimeter for rectangles. Another focuses on conversion from larger to smaller units of measure and suggests a focus on length, not area. A third standard explicitly names the use of rectangular arrays in understanding multiplication (as in Grade 3). The "geometric measurement" focus at this grade is angle and angular measure. Overall, given the strong attention to area measurement in Grade 3, it is somewhat surprising that there are no new objectives for area measurement in Grade 4.

• *Use place value understanding and properties of operations to perform multi-digit arithmetic [Number and Operations in Base Ten]*

> *5. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.*

Rectangular arrays and area models are recognized as useful means for making sense of multi-digit multiplication. Standard 6 (not stated here) makes similar mention of area models for quotients in the same numerical range.

• *Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.*

> *1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...*
>
> We could assume that students are expected to convert from one area unit to another in this grade even though there is not an area example in the actual statement.
>
> *3. Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*

We have included Standard 1 above because unit conversion is one important new issue in measurement addressed at this grade. Where in Grade 5, the focus is on converting from smaller units to larger, here in Grade 4 the focus is conversion from larger to smaller units—perhaps because the authors see such conversions as less challenging. But no explicit mention of area units is given in the statement's list of units or examples of specific unit conversions. This is a crucial omission because the conversion of area units do not obey the same logic as do length units. One square meter is 100 x 100 square centimeters, and this quadratic relationship is known area of difficulty for students—that relates directly to the challenge of understanding what the multiplication of lengths does to produce area measures discussed above. It seems a serious oversight not to call attention to this challenge here or in subsequent grades.

Standard 3 simply applies the understandings that have been targeted in Grade 3 to "real world" and "mathematical" problem contexts. What the author have in mind for "mathematical" problems in contrast to "real world" problems remains unclear.

Grade 5 (area)

The introduction (p. 33) focuses one of three critical areas on volume measurement; there is (again) no explicit discussion of issues of area measurement. One specific standard discusses rectangles with fraction side length in support of fraction multiplication. Another calls for the application of rectangular arrays in understanding multiplication of larger numbers. A third focuses on conversion, at this grade from smaller to larger units of measure. At this grade level, as before, the focus for unit conversion seems to be length; there are no explicit references to the conversion of units of area measure.

• *Perform operations with multi-digit whole numbers and with decimals to hundredths [Number and Operations in Base Ten]*

6. Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

This standard builds incrementally on related standards in Grade 4, endorsing the same three representations for multiplication and division.

• *Apply and extend previous understandings of multiplication and division to multiply and divide fractions [Number and Operations –Fractions]*

4b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

This standard also builds on a related standard in Grade 4 where the sides of rectangles have length measures of whole numbers. Here students must grapple with the need for and effects of fractional units of area in order to complete a successful tiling of the rectangle. Note that the discussion matches what was given in Grade 3: Counting fractional area units produces the same number as does multiplication of side lengths. No explanation of how this happens is offered. This advance seems difficult in cases where both sides are fractional length units and/or are "difficult fractions" (e.g., with odd denominators). Even relatively simple cases such as ¼ inch by ½ inch generates the challenge of understanding what the resulting area unit is (1 square inch) and how that whole unit is related to the product/area of the rectangle (1/8 of a square inch).

• *Convert like measurement units within a given measurement system.*

Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

As stated above, this standard extends the work on unit conversion to working from smaller units to larger. But as above, the single example is a length conversion so that challenge of converting area units remains invisible and unaddressed at this grade.

Grade 6 (area)

The introduction (p. 39) explicitly focuses on area measurement, but not as one of the four critical areas. These four address (1) rate and ratio, (2) more work with fractions and rational numbers and multiplication and division specifically, (3) variables and expressions in algebra, and (4) statistics, especially measures of central tendency. But the authors then append a 5th paragraph that appears to be an additional critical area. *Students in Grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine.* One plausible explanation of this "add on" statement is the authors' reluctance to include more than four critical areas at any grade. Their effectively doing just that is one indication that Grade 6 contains a comparatively larger proportion of new content than previous grades.

Specific standards address (1) using the relationship for the area of rectangles to determine, via composition and decomposition, the area of triangles and other quadrilaterals and (2) finding the surface area in the nets of three-dimensional shapes. Recall that the *Measurement and Data* domain disappears in Grade 6, so the most appropriate location for these standards is the *Geometry* domain.

• *Solve real-world and mathematical problems involving area, surface area, and volume [Geometry]*

> *1. Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.*

Here and in the introduction, the focus in determining the area for non-rectangular shapes is the most general one: Decompose and recompose into rectangular shapes whenever possible. Because of the generality of this single approach (relative to a list of different formulae that students may see no relation among), this focus has merit. But there is a subtle shift between this statement and the introduction on two issues: (1) the range of new shapes (beyond rectangles) whose area students should be able to determine and (2) whether students must learn formulas for computing the area of those shapes. The introduction explicitly states the expectation to develop (and justify!) formulas for the area of triangles and parallelograms, but Standard 1 offers a different list, including "special quadrilaterals" and "polygons." Though we cannot be sure, a sensible reading of these two statements would suggest a focus on right triangles, general triangles, and quadrilaterals. More fundamentally problematic is the absence of guidance for how these formulas should be developed. The authors may assume that reference to decomposition/recomposition into rectangles is sufficient conceptual ground for developing the formulas for triangles and parallelograms, but if so, that optimism does not seem warranted.

> *4. Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.*

This standard addresses the transformation of three-dimensional shapes into a representation of their two-dimensional faces. Though computing the sum of the areas of the faces of three-dimensional shape and seeing that sum as an area are realistic goals at this grade, the confusion of surface area with volume is the three-dimensional parallel to the confusion of perimeter and area. Teachers will need to provide support for students in speaking and thinking clearly about these two measures when they arise from the same shape.

Grade 7 (area)

The introduction gives explicit attention to area measurement in one of the four critical areas (p. 46). *Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in Grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes and right prisms.* New foci at this grade are (1) measurement in the circle (perimeter here and below but also area, below), (2) relationships of scale (that build on the work on rate and ratio in Grade 6), (3) informal geometric constructions, and (4) relating three-dimensional and two-dimensional shapes via cross-section. The relevant standards appear in the *Ratios and Proportional Relationships* domain and the *Geometry* domain.

• *Analyze proportional relationships and use them to solve real-world and mathematical problems [Ratios and Proportional Relationships]*

1. Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction $^{1/2}/_{1/4}$ *miles per hour, equivalently 2 miles per hour.*

This standard illustrates the notion of unit rate with an example of walking speed, but also refers to ratios of lengths and areas. This statement does little to clarify the authors' meaning for "rate" and "ratio," left unclear from Grade 6. Both terms appear in this standard. One important mathematical issue is left unaddressed: That the ratio of two lengths or two areas is a pure number (a "scalar") not a length or an area. For example, the ratio of "3 inches to 6 inches" may be expressed in a number of ways, including "one to two," "1:2," and "half as long." But none of these are themselves lengths.

• *Draw construct, and describe geometrical figures and describe the relationships between them (Geometry]*

1. Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Making accurate scale drawings requires the application of ratios to the actual lengths in a situation or object to reduce or increase them to desired size, e.g., in a drawing of a house to be constructed. The standard provides no guidance in carrying out this work, makes no explicit linkage to ratios, and makes no reference to common challenges that students face in work with scale (e.g., the common use of additive relationships in scaling up or down when ratios are multiplicative). It is also worth noting that when scale is applied correctly to length, areas scale up or down as a direct consequence.

• *Solve real-life and mathematical problems involving angle measure, area, surface area, and volume [Geometry]*

4. Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Standard 6 is a clear extension of work in previous grades, but Standard 4 represents quite a leap. The statement, that provides no guidance on the development of the formulae for the circumference and area of a circle, seems to suggest learning via memorization. If so, that is not a promising approach. The author could have appealed to decomposition/recomposition for the circle, but they chose not to, perhaps because of these methods typically involve some approximation (or a leap of faith) from what is measured and/or computed to π as an irrational number.

Grade 8 (area)

The introduction (p. 52) explicitly mentions the Pythagorean Theorem in one of three critical areas. *Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways.* The authors explicitly choose to frame this famous relationship in terms of area measurement (via decomposition) not length or distance, in the introduction at least.

• *Understand and apply the Pythagorean Theorem [Geometry]*

6. Explain a proof of the Pythagorean Theorem and its converse.

7. Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

8. Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

The authors closely relate understanding and using the relationship with being able to explain its proof, though it is not clear how rigorous or detailed an explanation is expected (Standard 6). They appear to shift focus from an area perspective in the introduction to a length perspective in the specific standard (7). Our perspective is that it is difficult to remove either the area or the length perspective from an understanding of this important relationship. Though demands for concise statement would not allow the authors to develop and clarify this point, the absence of any mention of it is problematic, for teachers and students.

VI. The Treatment of Volume Measurement in the CCSSM

Overview

The treatment of volume measurement in the CCSSM begins in Grade 5 when the measure is explicitly introduced. Preparation for volume measurement consists of (1) the exploration of three-dimensional shapes as part of the work in the *Geometry* domain beginning in Grade K and (2) two individual references to "liquid volume" in Grades 3 and 4. This treatment of volume represents a significant departure from current curricular practice—both at the level of specifications of written curricula in current state frameworks and written elementary curricula themselves. In these "curricula" (both state frameworks and textbooks), capacity is presented as a property of liquid-holding containers in Grade K and developed through the primary grades. Volume, typically defined as the count of cubes that fill boxes, is introduced somewhat later (e.g., Grades 2 and 3). Across the elementary years in many states and written curricula, the focus on capacity declines as attention to volume increases. The conceptual relationship between capacity and volume is not squarely and/or clearly addressed. With its nearly sole focus on volume, the CCSS-M avoids this confused and confusing treatment of two closely related quantities. In that sense, of the three spatial quantities (length, area, and volume), the document's treatment of volume measurement represents the greatest departure from current educational practice. In this case, we view this change positively.

Where the document's treatment is both clearer and more temporally focused that current practice (which extends the development of capacity + volume from Grade K to Grade 8), that treatment is not without limitations and problems. We cite three.

• Where the confusing term "capacity" has completely disappeared, the confusing term "liquid volume" does appear. The document does not provide sufficient guidance in how teachers in understanding "liquid volume" relative to "volume" as defined as the count of cubic units. This problem is exacerbated by the vague definition of volume presented in Grade 5.

• The focused presentation of volume measurement in Grade 5 is undermined by a weak definition of the quantity that fails to clearly identify <u>which attribute</u> of three-dimensional shapes is volume. There is no clear statement that volume is the <u>amount of space</u> enclosed in a three-dimensional shape. Equally problematic is the failure to extend the discussion of the conceptual properties of units that the document presents for length equally well for volume. In this sense, the problems of weak presentation of conceptual issues noted above for area is also true for volume.

• The preparation for volume measurement involves the exploration and analysis of three-dimensional shapes, beginning in Grade K and continuing through Grade 2. These standards do not always clarify how students' work with three-dimensional shapes should build and deepen across the primary grades.

Volume Measurement Standards, Grade-by-Grade

Unless otherwise noted, all standards listed below come from the *Measurement and Data* domain. Content stated literally in the CCSSM is given in italics; our interpretive statements are in plain text. The term "introduction" refers to (a) the small number of "critical areas" and (b) the list of brief standard statements that are offered at the beginning of grade-specific section of the document. In general, because many of the issues that arise in the standard listed below have been discussed in previous sections, our consideration of them here is briefer.

Kindergarten (volume)
As we noted for area in Section V, the introduction (p. 9) focuses one of three critical areas on the description of two-and three-dimensional shapes. *Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.*

No mention of volume or liquid volume appear in either the *Measurement & Data* standards or the *Geometry* standards. Three *Geometry* standards focus on the exploration and construction of two- and three-dimensional shapes; Standard 4 explicitly names work with three-dimensional figures.
• *Analyze, compare, create, and compose shapes [Geometry]*
4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
5. Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
6. Compose simple shapes to form larger shapes. For example, "Can you join these two triangles with full sides touching to make a rectangle?"

Grade 1 (volume)
The measurement focus at Grade 1 is on length. But the introduction (p. 13) again devotes one of four critical areas to the composition and decomposition of two-dimensional and three-dimensional shapes, as we saw for area. *Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.* Where there is clear focus and attention to three-dimensional geometry here, it is unclear how work at Grade 1 differs from and builds on work at Grade K. With respect to volume measurement, it is unclear what developing "the background for measurement" means here for three-dimensional shapes, beyond the general idea that shape matters for area and volume measurement.

Measurement & Data standards at Grade 1 concern length measurement, time, and representing data. One *Geometry* standard addresses the analysis of three-dimensional shapes with particular attention to composition of complex shapes from simpler ones. To some extent, the focus on composition addresses the concern raised above, yet the range and purposes of composition remain unclear.
• *Reason with shapes and their attributes. [Geometry]*

2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. (Students do not need to learn formal names such as "right rectangular prism").

Grade 2 (volume)

The Grade 2 measurement focus remains on length, but the introduction (p. 17) again includes one critical area focusing on the exploration of two- and three-dimensional shapes. *Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.* Here attention is given to analysis, and the analytic focus is on particular geometric features of shapes (side and angles). But that focus is clearer for two- than it is for three- dimensional shapes. There is less guidance for the analysis of three-dimensional shapes via composition and decomposition.

The focus of *Measurement & Data* standards is on length measurement, money, and representing data. A single *Geometry* standard focuses on recognizing and drawing shapes; its focus in three dimensions is on cubes and their defining characteristic of all equal faces.

• *Reason with shapes and their attributes.[Geometry]*

1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[Footnote: Sizes are compared directly or visually, not compared by measuring.] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

Grade 3 (volume)

The Grade 3 measurement focus is on area; no explicit attention is given to volume in the introduction (p. 21). One critical area focuses on area measurement and another on two-dimensional geometry.

But one *Measurement & Data* standard makes reference to "liquid volume" as a quantity [emphasis added]. The term "geometric measurement" is introduced in Grade 3 and extensively developed, but references are restricted to two dimensions.

• *Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.*

2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l) (Excludes compound units such as cm^3 and finding the geometric volume of a container). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem. [Footnote: Excludes compound units such as cm^3 and finding the geometric volume of a container.]

To our knowledge, this reference and another below in Grade 4 are the only references to what is presented in current elementary curricula as "capacity." However, no attention is given to helping the reader understand "liquid volume" in relation to "volume" (introduced and defined later in Grade 5). That both liquid volume and solid objects fill space is not stated. "Liquid volume" shifts attention away from capacity as a property of containers and toward a quantity that takes up three-dimensional space. But that conception does not immediately square with the definition of volume introduced in Grade 5.

Grade 4 (volume)
The introduction (p. 27) contains no reference to measurement. The Grade 4 measurement focus is on unit conversion (larger to smaller units) and angle measure, with no explicit attention to volume. Units of (liquid) volume—liters and milliliters—are included in the standard addressing unit conversion, though the emphasis is on length units. A second standard includes "liquid volumes" as one quantity appearing in applied ("word") problems involving unit conversions. Emphasis has been added below to locate these references in the two standards.

• *Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.*

> *1. Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36),*
>
> *2. Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.*

Grade 5 (volume)
One of three critical areas named in the introduction focuses on volume measurement (p. 33). *Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real world and mathematical problems.* This opening statement roughly parallels the introduction to area measure at Grade 3. The initial statement (what volume is as opposed to how it is computed) provides no clear reference to the amount of three-dimensional space as the specific measurable attribute that volume measures. As for area, the introduction emphasizes the discrete meaning of volume—the number of cubes required to fill a space.

One long *Measurement & Data* cluster of standards provides more detail on the concepts and procedures of volume measurement to be developed in Grade 5, now under the heading of "geometric measurement." (In Grade 3, this term was restricted to measurement in two dimensions and area measurement specifically; in Grade 4, it included to angle and angular measurement.)

• *Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.*

3. Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

(a) A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

(b) A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

4. Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5. Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

(a) Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

(b) Apply the formulas $V = l \times w \times h$ *and* $V = b \times h$ *for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.*

(c) Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

As a careful reading of the document makes clear (see Grade 3; geometric measurement), the structure and content of this cluster nearly identically parallels the discussion of area measurement at Grade 3. The only significant change between the two clusters is the shift from two-dimensional space at Grade 3 to three-dimensional space here. Since many of the same issues we addressed for area return here, our discussion of this cluster is briefer than it was for area.

As in the introduction, Standard 3 contains no reference to the amount of three-dimensional space that will be filled with cubic units. As was the case for area, the only recognized units of volume are standard units—cubes with edge length of a standard unit of length. "Filling" suggests the tiling rather than iterating of cubic units; the equivalence between filling a prism with cubic units and multiplying its height by the area of its base and between the latter and multiplying length by width by height is suggested as the justification for these computational formulas; and volume measurement is asserted to be additive in nature. But there is no discussion of the inverse relationship between the size of volume units and the number required to fill any given space. The *Measurement & Data* standard that extends unit conversion from smaller to larger units includes no mention of volume units. More specifically, no mention is given here (or in later grades) to the fact that unit conversion among volume units often involves non-linear scaling (e.g., 1 cubic yard = 27 cubic feet).

<u>Grade 6 (volume)</u>

The introduction (p. 39) states four critical areas, none of which explicitly mention three-dimensional shapes or volume measurement. However, as noted for area, the Grade 6 introduction is much longer than those for prior grades and contains what appears to be a fifth (but unnumbered) critical area. That paragraph states (emphasis below has been to facilitate locating the references to volume): *Students in Grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and* <u>*volume*</u>. *They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine. They reason about right rectangular prisms with fractional side lengths to extend formulas for the* <u>*volume*</u> *of a right rectangular prism to fractional side lengths. They prepare for work on scale drawings and constructions in Grade 7 by drawing polygons in the coordinate plane.* One clear addition in this paragraph is the attention to work with the volume of prisms with fractional side lengths.

Specific standards address volume measurement content in the domains of *Ratios & Proportional Relationships* (connecting work on unit conversion to ratios, without explicitly mentioning volume), *Expressions and Equations*, and *Geometry* (where work with volume formulae are extended to lengths measured in fraction units in two standards). The disappearance of the *Measurement & Data* domain from the middle grades (6-8) forces the placement of metric content in the *Geometry,* even though that choice voids the prior distinction between descriptive geometry and metric (numerical) measurement.

• *Understand ratio concepts and use ratio reasoning to solve problems. [Ratios & Proportional Relationships]*

3. Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

In one sense, there is no difference between using "ratio reasoning to convert measurement units." But for volume units, the ratio (a cubic relationship) has proven difficult for middle school students to grasp. The document does not name this specific difficulty.

• *Apply and extend previous understandings of arithmetic to algebraic expressions. [Expressions and Equations]*

2. Write, read, and evaluate expressions in which letters stand for numbers. c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ *and* $A = 6 s^2$ *to find the volume and surface area of a cube with sides of length* $s = 1/2$.

• *Solve real-world and mathematical problems involving area, surface area, and volume. [Geometry]*

2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = l\ w\ h$ *and* $V = b\ h$ *to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.*

This statement parallels similar ones for area (see Section V). But where area units that "spill over" the boundaries of two-dimensional shapes are not problematic and may even support the determination of fraction units of area, unit cubes will not "fill" a right rectangular prism when the edge length is not a multiple of the cubic unit—at least in any physical ("packing") sense. This is simply a misleading statement.

Grade 7 (volume)
One of four critical areas in the introduction (p. 46) addresses a mixture of two- and three-dimensional geometric issues and area (Section V) and volume measurement issues. The volume measurement foci are on relating shape in two- and three-dimensional by focusing on cross-sections and on problem solving. *Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in Grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes and right prisms.* As before, it is not clear how the authors see the distinction between "real-world" and "mathematical" problems in the context of measurement.

• *Draw construct, and describe geometrical figures and describe the relationships between them [Geometry]*

   3. Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

• *Solve real-life and mathematical problems involving angle measure, area, surface area, and volume. [Geometry]*

   6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Grade 8 (volume)
The measurement focus at Grade 8 is primarily two-dimensional in focus—on congruence, similarity, and the Pythagorean Theorem. But the introduction (p. 52) includes one sentence in one of three critical areas that concerns volume measurement: *Students complete their work on volume by solving problems involving cones, cylinders, and spheres.* It is surprising that the authors speak of "completing work on volume" by learning formulas for three new geometric shapes.

• *Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. [Geometry]*

   9. Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

As we have seen for area measurement (i.e., the Pythagorean Theorem), the document calls for knowing the formulas for the volume of cones, cylinders, and spheres, without specifying how students should come to know. The formulas for three shapes are not all of equal difficulty—the volume of cylinder is accessible to the "base time height" logic discussed at prior grades. Cones and spheres, by contrast, seem to make greater demands. The absence of specific guidance here for learning these formulas is consistent with, if not suggestive of learning by memorization.